Partition functions of the two-dimensional Ashkin-Teller model on the critical line

1987 J. Phys. A: Math. Gen. 20 L1127

(http://iopscience.iop.org/0305-4470/20/16/016)

J. Phys. A: Math. Gen. **20** (1987) L1127–L1133. Printed in the UK

LETTER TO THE EDITOR

Partition functions of the two-dimensional Ashkin–Teller model on the critical line

H Saleur
Service de Physique Théorique, CEN-Saclay, 91191 Gif-sur-Yvette Cedex, France

Received 7 August 1987

Abstract. By mapping the Ashkin–Teller model onto a scalar free-field theory we derive the continuum limit of the partition functions in various finite geometries. We justify expressions which have been conjectured recently, explaining in particular why the free field lives on an orbifold and why the spin should be described by a twist operator.

The Ashkin–Teller (AT) model [1] appears of great interest in the light of conformal invariance. It presents a critical line, corresponding to a $c=1$ central charge [2]. Numerical studies [3] have suggested that the spectrum of the transfer matrix for free boundary conditions (BC) is $N=2$ supersymmetric [4] at three particular points of this line, while the case of periodic or antiperiodic BC has led to controversial results [3, 5, 6]. Recently, using a reasonable ansatz and some known properties at the Ising decoupling point [1], Yang [5] has obtained expressions for the continuum limit of the AT partition function in various finite geometries. He observed that the results could be interpreted in terms of a scalar free-field theory compactified on a circle [7] with a possible identification of φ with $-\varphi$ (orbifold). In this letter, we derive these partition functions, using Coulomb-gas techniques which are well known in statistical mechanics [8, 9]. We interpret physically the free-field formulation, explaining in particular why the AT spin translates into a twist operator [5, 10].

The Ashkin–Teller model [1] consists of two Ising models coupled by a four-spin interaction, with action

$$\mathscr{A} = -\sum_{\langle jk\rangle} K_2(S_jS_k + t_jt_k) - K_4S_jS_kt_jt_k \tag{1}$$

where $\langle jk\rangle$ denotes nearest neighbours of the square lattice $\mathscr{L}$, and $S,t = \pm 1$. The key of our study will be the reformulation into a solid-on-solid (SOS) surface model, along the lines of [8]. First it is convenient to rewrite (1) (using the invariance of the partition function $\mathscr{Z}^{\mathrm{AT}}$ under $t \to St$) as

$$\mathscr{A} = -\sum_{\langle jk\rangle} K_2S_jS_k(1+t_jt_k) - K_4t_jt_k \tag{2}$$

and $\mathscr{Z}^{\mathrm{AT}}$ is

$$\mathscr{Z}^{\mathrm{AT}} = \sum_{\{S,t\}} \prod_{\langle jk\rangle} \exp(K_4t_jt_k)\cosh(K_2+K_2t_jt_k)[1+S_jS_k\tanh(K_2+K_2t_jt_k)]. \tag{3}$$

A $\{t\}$ configuration can be represented by putting bonds on the dual lattice $\mathscr{D}$ which separates two sites with opposite t, as in an Ising low-temperature expansion [11].

The product of square brackets can then be expanded as in an Ising high-temperature expansion [11], with a bond on $\mathscr{L}$ each time the S_jS_k term is taken. Summing over all t and S gives

$$\mathscr{Z}^{\mathrm{AT}} = 2^N \exp(2NK_4)(\cosh 2K_2)^{2N} \sum_{\mathrm{graphs}} (\tanh 2K_2)^l \left(\frac{\exp(-2K_4)}{\cosh 2K_2}\right)^d \tag{4}$$

where N is the total number of sites, the graphs are formed by polygons on $\mathscr{L}$ and $\mathscr{D}$ with an even number of bonds attached to any point, the total numbers of bonds on each lattice being respectively l and d (see figure 1). If a given bond is present on $\mathscr{D}$, the product of the two corresponding spins is -1 and the tanh in (3) is zero. Thus the polygons on $\mathscr{L}$ and $\mathscr{D}$ do not intersect. The model presents a critical line given by the self-duality condition $\exp(-2K_4) = \sinh 2K_2$, which terminates at $\coth 2K_2 = 2$. The graphs in (4) can be alternatively represented by six-vertex [11] configurations (figure 2) on the surrounding lattice $\mathscr{S}$ (another square lattice, the vertices of which are the midpoints of the edges of $\mathscr{L}$, see figure 1). A bond on $\mathscr{L}$ or $\mathscr{D}$ is associated to a vertex of type $1, \ldots, 4$, such that arrows are reflected by it, with a net non-zero polarisation (figure 2). Edges with no bond are associated to vertices of type 5 or 6.

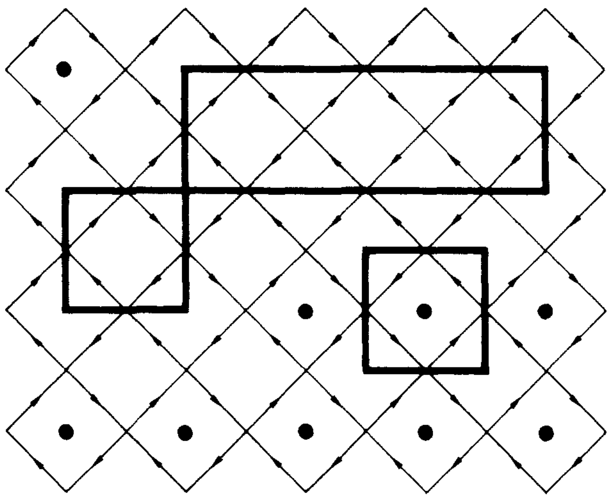

Figure 1. A graph in the high-temperature expansion involving a polygon on $\mathscr{L}$ (whose sites are indicated by full circles) and a polygon on $\mathscr{D}$. The corresponding six-vertex configuration is indicated.

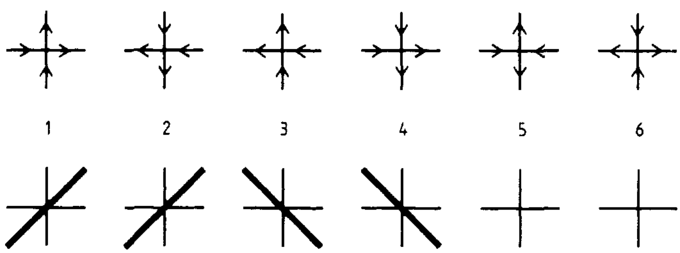

Figure 2. Arrow configurations in the six-vertex model. Vertices of type $1, \ldots, 4$ are associated to bonds on $\mathscr{L}$ or $\mathscr{D}$, vertices of type 5, 6 to edges with no bonds.

Once a possible vertex is chosen somewhere, the whole correspondence follows by induction. A given configuration of bonds is thus associated with a configuration of the six-vertex model (defined up to a reversal of all arrows) and vice versa. Along the critical line, this six-vertex model reduces to an F model [11] with Boltzman weights $W_1 = \ldots = W_4 = 1$, $W_5 = W_6 = \coth 2K_2$. Now the F model can be transformed into an SOS model [12] by introducing height variables φ on the faces of $\mathscr{S}$, such that two neighbouring φ differ by $\pm\varphi_0$, the highest being on the left of each arrow. It is finally argued [8, 9] that this SOS model renormalises onto a Gaussian model, with the free-field action

$$\mathscr{A}_1 = \frac{g}{4\pi}\int |\nabla\varphi|^2 \mathrm{d}^2 x. \tag{5}$$

For the standard choice $\varphi_0 = \frac{1}{2}\pi$, the value of the coupling constant is [8, 13]

$$g = (8/\pi)\sin^{-1}(\tfrac{1}{2}\coth 2K_2). \tag{6}$$

The operators of the theory (5) which have been studied most in statistical mechanics are the electric operators (vertex operators in the language of strings), i.e. exponentials of the free field $\mathscr{O}_e = \exp ie\varphi$ (e is called an electric charge), and the dual magnetic operators $\mathscr{O}_m$, the correlation function of which is obtained by imposing a discontinuity of $2m\pi$ for the field φ along a line connecting two points (m is called a magnetic charge). Combining these two kind of operators one gets a more general object $\mathscr{O}_{em}$ with dimension and spin [8]

$$x_{em} = e^2/2g + gm^2/2 \qquad s_{em} = em. \tag{7}$$

It is well known, for instance [8, 9], that the thermal exponent of the AT model is given by $x_T = x_{20} = 2/g$, the polarisation exponent by $x_p = x_{10} = 1/2g$ and the anisotropy exponent by $x_{CR} = x_{01} = g/2$. These depend on the temperature through (6) and (7). In opposition to this, the magnetic exponent x_H cannot be obtained by (7) and it has been conjectured for a long time [9] that it remains constant along the critical line, with the Ising value $x_H = \frac{1}{8}$. Several authors [5, 10] have noticed that this feature is characteristic of the twist operators $\mathscr{O}_T$ which have been much studied in string theory; their correlation function is defined by imposing a twist $\varphi \to -\varphi$ of the field φ along a line connecting two points, and they play a fundamental role in the construction of field theories on an orbifold [14].

So far we have discussed the Ashkin–Teller model on an infinite plane. In a finite geometry, the boundary conditions generate various constraints on the successive transformations described above, resulting in a modified free-field theory (5) as we already demonstrated in [15]. Consider the case of a torus defined by two periods ω_1, ω_2. Then the above graph formulation (4) is still valid, but an asymmetry appears between $\mathscr{L}$ and $\mathscr{D}$: a closed path homotopic to ω_1 or ω_2 has to cross an even number of bonds on $\mathscr{D}$ since these correspond to a spin flip, while the number of crossed bonds on $\mathscr{L}$ can be arbitrary. We shall first study the case where both these numbers are even; the reformulation as an F model is then possible in the same way as before. Now, since variables φ are associated locally to vertex configurations, they cannot be defined in a consistent way [15]: describing a closed path along $\omega_1(\omega_2)$ the height varies by a finite amount $\delta_1\varphi(\delta_2\varphi)$. The restriction of an even number of crossed bonds on $\mathscr{L}$ and $\mathscr{D}$ translates then into the condition $\delta_1\varphi = 2\pi m (\delta_2\varphi = 2\pi m')$, $m(m') \in \mathbb{Z}$. For given m, m', the continuum limit is the frustrated partition function introduced in [15]

$$Z_{mm'} = \int_{\substack{\delta_1\varphi = 2\pi m \\ \delta_2\varphi = 2\pi m'}} [\mathrm{D}\varphi] \exp(-\mathscr{A}_1) \tag{8}$$

which is readily evaluated using the classical solution (such that $\Delta\varphi_{\text{class}} = 0$)

$$Z_{m,m'} = Z_1 \exp\left(-\pi g \frac{m'^2 + m^2(\tau_R^2 + \tau_I^2) - 2m'm\tau_R\tau_I}{\tau_I}\right). \tag{9}$$

In this expression, $\tau = \omega_2/\omega_1 = \tau_R + i\tau_I$ is the modular ratio and Z_1 is the partition function of the (periodic) free field (5)

$$Z_1 = \frac{\sqrt{g}}{\tau_I^{1/2}\eta(q)\eta(\bar{q})} \tag{10}$$

where η is Dedekind's function

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \qquad q = \exp(2i\pi\tau).$$

Summing over m, m' one gets the Coulombic partition function of [5, 15]

$$Z_c[g, 1] \equiv \sum_{mm' \in \mathbb{Z}} Z_{mm'} = \frac{1}{\eta\bar{\eta}} \sum_{e,m\in\mathbb{Z}} q^{(e\sqrt{g} + m\sqrt{g})^2/4} \bar{q}^{(e\sqrt{g} - m\sqrt{g})^2/4} \tag{11}$$

(this last equality being obtained by the Poisson formula) which is clearly modular invariant. The small-q behaviour $Z \sim (q\bar{q})^{-c/24}$ gives the central charge $c = 1$, while the other terms $q^{h_{em}}\bar{q}^{\bar{h}_{em}}$ reproduce the whole spectrum (7) ($x_{em} = h_{em} + \bar{h}_{em}$, $s_{em} = h_{em} - \bar{h}_{em}$) with $e, m \in \mathbb{Z}$. Note this is not the continuum limit of the F-model partition function. In the latter, frustration multiples of π are also allowed so that [15]

$$\mathscr{Z}^F \to Z_c[g, \tfrac{1}{2}] \equiv \frac{1}{\eta\bar{\eta}} \sum_{\substack{e\in 2\mathbb{Z} \\ m\in\mathbb{Z}/2}} q^{h_{em}}\bar{q}^{\bar{h}_{em}}. \tag{12}$$

The continuum limit of the AT partition function will be obtained by adding to (11) the contribution of graphs where a closed path crosses an odd number of $\mathscr{L}$ bonds and an even number of $\mathscr{D}$ bonds. This can happen for a path homotopic either to ω_1, or to ω_2, or to ω_1 plus ω_2 and there is a splitting in three different sectors. For a given sector the duality invariance ensures one gets the same result by exchanging the role of $\mathscr{L}$ and $\mathscr{D}$, so it is equivalent to consider the case where an odd number of bonds on $\mathscr{L}$ plus $\mathscr{D}$ is crossed. This corresponds in turn to an odd number of vertices of type 5,6 crossed, i.e. to antiperiodic BC for the F model. We now discuss the SOS equivalence by considering for instance the case of antiperiodic BC in the ω_2 direction (broken line in figure 3) and periodic BC in the ω_1 direction. For an arbitrary column there is a discontinuity $\delta_2\varphi$ which is multiple of π, and it is always possible to choose the origins of the heights so one goes from φ to $-\varphi$ by crossing the broken line (figure 3). Consider now the neighbouring columns. In the case of periodic BC for the F model, the vertex rules would ensure that $\delta_2\varphi$ is conserved so the heights along the broken line would become $\varphi \pm \frac{1}{2}\pi$, $-\varphi \pm \frac{1}{2}\pi$, while here the antiperiodic BC change $\delta_2\varphi$ into $\delta_2\varphi \pm \pi$, thus the heights on both sides of the broken line become $\varphi \pm \frac{1}{2}\pi$, $-(\varphi \pm \frac{1}{2}\pi)$ and the twist is conserved. By winding in this way around the whole torus along ω_1

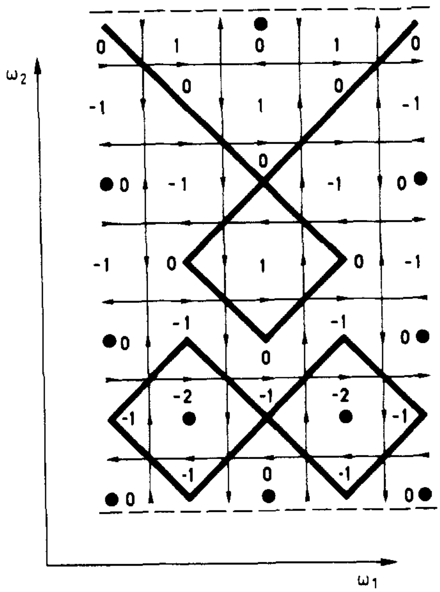

Figure 3. A graph for which the number of $\mathscr{L}$ bonds crossed by a closed path along ω_1 (ω_2) is even (odd). It translates into an F model with periodic (antiperiodic) BC in the ω_1 (ω_2) direction, the latter being indicated by the broken line. This in turn corresponds to an SOS model with periodic (twisted) BC in the ω_1 (ω_2) direction. Heights φ are indicated in multiples of $\pi/2$.

one gets $\delta_1\varphi = 0$ since $\delta_1\varphi \neq 0$ would be incompatible with the existence of the horizontal twist. Thus antiperiodicity in the F model translates into a twist for the SOS model. The same is valid for other sectors. In the continuum limit one gets

$$Z_{\alpha\beta} = \int_{\substack{\varphi(z+\omega_1)=\exp(2\mathrm{i}\pi\alpha)\varphi(z)\\ \varphi(z+\omega_2)=\exp(2\mathrm{i}\pi\beta)\varphi(z)}} [\mathrm{d}\varphi]\exp(-\mathscr{A}_1) \qquad (\alpha,\beta=0 \text{ or } \tfrac{1}{2};\ (\alpha,\beta)\neq(0,0)) \tag{13}$$

which has been calculated in [16],

$$Z_{\alpha\beta} = q^{[6\alpha(1-\alpha)-1]/24}\prod_{n=0}^{\infty}(1-\exp(2\mathrm{i}\pi\beta)q^{n+\alpha})^{-1/2}$$
$$\times(1-\exp(-2\mathrm{i}\pi\beta)q^{n+1-\alpha})^{-1/2}\times\text{complex conjugate.} \tag{14}$$

Since there is no zero mode preventing the rescaling of φ, this does not depend on the coupling g and remains constant along the whole critical line. Summing over $(\alpha,\beta)\neq(0,0)$ one gets the other contribution to the AT partition function as $Z_{1/2,0}+Z_{0,1/2}+Z_{1/2,1/2}$ which is modular invariant. $Z_{1/2,0}$ and $Z_{1/2,1/2}$ have the small-q behaviour $\sim(q\bar{q})^{-1/24}(q\bar{q})^{1/16}$ thus reproducing the dimension of the spin operator. This is natural since the graph expansion of the spin correlation function $\langle S_j \cdot S_k\rangle$ involves vertices with an odd number of legs in j and k, and thus an odd number of crossed bonds as explained above. $Z_{0,1/2}$ contains a marginal operator with dimension $x=2$ constant along the critical line [17]. The complete AT partition function on the torus is obtained by combining (11) and (14). The relative normalisations cannot be fixed by the microscopic approach (since there is a zero-mode subtraction in (11) only) but are easily obtained by demanding the identity (spin) operators to be non-(twice) degenerate resulting in

$$\mathscr{Z}^{\mathrm{AT}} \to \tfrac{1}{2} Z_{\mathrm{c}}[g, 1] + Z_{1/2,0} + Z_{0,1/2} + Z_{1/2,1/2}. \tag{15}$$

This expression was also obtained by Yang who conjectured $\mathscr{Z}^{\mathrm{AT}}$ could be written as (11) plus a constant term, and identified the latter by using the fact that at $g=2$ ($K_4=0$), the AT model decouples into two Ising models and thus $\mathscr{Z}^{\mathrm{AT}} = (\mathscr{Z}^{\mathrm{Ising}})^2$. He also noticed each term of (15) had an interpretation in terms of a scalar free-field theory compactified on a circle. We have given here a microscopic justification of these different steps, explaining in particular why the spin operator in the AT model should translate into the twist operator in the Gaussian free field. Equation (15) can also be written

$$\mathscr{Z}^{\mathrm{AT}} \to \tfrac{1}{2}\{Z_{\mathrm{c}}[g, 1] - Z_{\mathrm{c}}[1, 1]\} + Z_{\mathrm{c}}[4,1]. \tag{16}$$

If $g=1$ this gives $Z_{\mathrm{c}}[4, 1]$, corresponding to the XY-model partition function at the Kosterlitz-Thouless point obtained in [15]. Similarly, for $g=4$ one gets $\{3Z_{\mathrm{c}}[4, 1] - Z_{\mathrm{c}}[1, 1]\}/2$, which is the four-state Potts model partition function [15].

We can now generalise these results. Consider a toroidal geometry with antiperiodic BC for the spins S and t in the ω_1 direction. This is equivalent to putting a negative coupling K_2 for each bond of $\mathscr{L}$ crossing a line parallel to ω_2 in the graph expansion. Following the preceding derivation we see that $Z_{0,1/2}$ and $Z_{1/2,1/2}$ contribute now with a minus sign since they are associated with an odd number of bonds crossing the line. Thus

$$\mathscr{Z}^{\mathrm{AT}}_{\mathrm{AP}} \to \tfrac{1}{2} Z_{\mathrm{c}}[g,1] + Z_{1/2,0} - Z_{0,1/2} - Z_{1/2,1/2} \tag{17}$$

and in the same way

$$\begin{aligned} \mathscr{Z}^{\mathrm{AT}}_{\mathrm{PA}} &\to \tfrac{1}{2} Z_{\mathrm{c}}[g, 1] - Z_{1/2,0} + Z_{0,1/2} - Z_{1/2,1/2} \\ \mathscr{Z}^{\mathrm{AT}}_{\mathrm{AA}} &\to \tfrac{1}{2} Z_{\mathrm{c}}[g, 1] - Z_{1/2,0} - Z_{0,1/2} + Z_{1/2,1/2}. \end{aligned} \tag{18}$$

These results were also obtained by Yang [5] assuming the Gaussian operators $\mathscr{O}_{em}$ were even and the magnetisation operators odd under spin reversal, which we established here.

Finally we consider a rectangle of sides $x=L$, $y=T$ with free (periodic) BC in the x (y) direction. Then the AT model translates into a SOS model with φ constant on both sides parallel to y. Thus there are no discontinuities in the y direction and any discontinuity multiple of π in the x direction (there is no more restriction on the parity of the number of crossed bonds in this case). Using

$$\int\limits_{\substack{\varphi(0,y)=\varphi(L,y)=0 \\ \varphi(z+T)=\varphi(z)}} [D\varphi] \exp(-\mathscr{A}_1) = \frac{1}{\eta(q^{1/2})} \qquad q=\exp(-2\pi T/L) \tag{19}$$

and the classical solution $\varphi_{\mathrm{class}} = \pi m x/L$ one gets

$$\mathscr{Z}^{\mathrm{AT}}_{\mathrm{FP}} \to \frac{1}{\eta(q^{1/2})} \sum_{m=0}^{\infty} \exp\left(-\pi \frac{g}{4} \frac{T}{L} m^2\right) = \frac{1}{\eta(q^{1/2})} \sum_{m=0}^{\infty} q^{gm^2/8}. \tag{20}$$

This was also obtained by Yang [5] who assumed $\mathscr{Z}^{\mathrm{AT}}_{\mathrm{FP}}$ could be described by a chiral Gaussian theory, and then used the decoupling point $g=2$ to determine the radius of compactification.

I thank P di Francesco and J B Zuber for many discussions.

References

[1] Ashkin J and Teller E 1943 *Phys. Rev.* **64** 178
[2] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl. Phys.* B **241** 333
[3] von Gehlen G and Rittenberg V 1986 *J. Phys. A: Math. Gen.* **19** L1039
Baake M, von Gehlen G and Rittenberg V 1987 *J. Phys. A: Math. Gen.* **20** L479, L487
[4] Boucher, W, Friedan D and Kent A 1986 *Phys. Lett.* **172B** 316
di Vecchia P, Petersen J L, Yu M and Zheng H B 1986 *Phys. Lett.* **174B** 280
[5] Yang S K 1987 *Nucl. Phys.* B **285** 183
[6] Yang S K and Zheng H B 1987 *Nucl. Phys.* B **285** 410
[7] Gross D J, Harvey J A, Martinec E and Rohm R 1985 *Nucl. Phys.* B **256** 253
[8] Nienhuis B 1984 *J. Stat. Phys.* **34** 731
[9] Kadanoff L P and Brown A C 1979 *Ann. Phys., NY* **121** 318
Knops H J F 1980 *Ann. Phys., NY* **128** 448
Kohmoto M, den Nijs M and Kadanoff L P 1981 *Phys. Rev.* B **24** 5229
[10] Friedan D and Shenker S 1986 unpublished
[11] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
[12] von Beijeren H 1977 *Phys. Rev. Lett.* **38** 993
[13] Black J L and Emery V J 1981 *Phys. Rev.* B **23** 429
[14] Dixon L, Harvey J, Vafa C and Witten E 1985 *Nucl. Phys.* B **261** 620
Dixon L, Friedan D, Martinec E and Shenker S 1987 *Nucl. Phys.* B **282** 13
[15] di Francesco P, Saleur H and Zuber J B 1987 *Nucl. Phys.* B **285** 454
[16] Itzykson C and Zuber J B 1986 *Nucl. Phys.* B **275** 580
[17] Kadanoff L P and Wegner F J 1971 *Phys. Rev.* B **4** 3989